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TRANSMISSION MATRIX REPRESENTATION OF EXHAUST SYSTEM ACOUSTIC CHARACTERISTICS

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The prediction of the propagation of sound in the hot gas flowing along an internal combustion (IC) engine exhaust duct is a common practical problem. Associated with the flow temperature $T(x)$, where x is the axial displacement along the duct and, say, one-dimensional motion or plane wave propagation, there will be a corresponding axial velocity $u(x, t)$, density $\rho(x, t)$ pressure $p(x, t)$ and entropy $s(x, t)$, all having both mean and fluctuating ("acoustic") parts. Such exhaust systems normally consist of a sequence of uniform pipes connecting area discontinuities such as expansions, orifices, branches and so on, which extend from the source of excitation to the final open termination. Each section or element of such systems can then be regarded as being excited by a preceding element and terminated by its successor.

The acoustic transfer characteristics of each element are often represented by an appropriate transfer matrix based on the approximate analogy that can be drawn between their acoustic behaviour and that of some corresponding electrical network. However, practical experience [1] has demonstrated that such acoustic analogies with network theory may not always provide valid predictions, and thus should only be adopted after appropriate consideration of their physical realism.

Acoustic plane wave transmission across an individual element or any sequence of elements is illustrated in Figure 1. Here, S represents the source of excitation for the

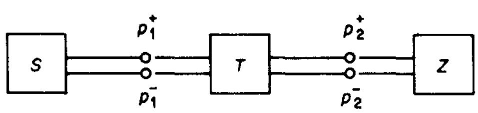

Figure 1. Plane wave acoustic transmission across an element.

element or sequence T, while Z represents the acoustic impedance at its termination. For isentropic plane acoustic wave propagation across the element, the complex amplitude† p^+ of the positively travelling and p^- of the negatively travelling wave components are related to the corresponding fluctuating acoustic pressure p and acoustic velocity u by

$$p = p^+ + p^-, \qquad \rho_0 c_0 u = p^+ - p^-, \tag{1, 2}$$

where ρ_0 and c_0 are respectively the undisturbed density and sound speed in the medium and p, u, p^+ and p^- are all functions of position x.

It has been demontrated [2] that equation (1) is valid irespective of the existence or otherwise of a steady mean flow velocity u_0, while equation (2) is similarly valid only as long as wave attenuation and dispersion by viscothermal actions remain negligible. The wave components $p_1^{\pm}$ on the source side of T can be related to $p_2^{\pm}$ on the load side by

$$p_1^+ = T_{11} p_2^+ + T_{12} p_2^-, \qquad p_1^- = T_{21} p_2^+ + T_{22} p_2^-, \tag{3, 4}$$

† Simple harmonic motion is assumed henceforth, with a common implicit time dependence factor $e^{i\omega t}$, where $\omega(=2\pi f)$ is the radian frequency, for all acoustic pressure and velocity quantities.

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where T_{11}, T_{12}, T_{21} and T_{22} are the four elements of the scattering transmission matrix $[T]$ defining the transfer. The complex values of these four elements are all functions of the geometry of the system element, of the undisturbed values of the density ρ_0 and sound speed c_0, of the frequency of excitation, f, of the mean flow Mach number $M = u_0/c_0$, but remain independent of the value of Z. This last fact provides an experimental or analytical method for establishing the values of the matrix elements.

Obviously, equations (1) and (2) imply that the fluctuating acoustic pressure p_1 and velocity u_1 complex amplitudes on the source side of T can be related to p_2 and u_2 on the load side in terms of the corresponding impedance transfer matrix $[t]$, by expressions analogous to equations (3) and (4). The four elements of $[t]$ exhibit a similar independence of Z and are also functions of geometry, ρ_0, c_0, f and M. The elements of the scattering matrix $[T]$ are all dimensionless combinations of the relevant physical parameters, while those of $[t]$ are not all similarly dimensionless. The relative difficulty of measuring the acoustic particle velocity, u, compared with measurement of the fluctuating pressure, p, combined with the restricted validity of equation (2), suggest that the scattering matrix $[T]$ provides more robust descriptions of the wave transfer than does the impedance matrix $[t]$.

To return to Figure 1, acoustic wave transmission across the system element T can also be expressed in terms of the transmission coefficients

$$T_i = p_1^+/p_2^+, \qquad T_r = p_1^-/p_2^-, \tag{5a, b}$$

together with the reflection coefficient, r, where

$$r = p_2^-/p_2^+, \qquad \text{or} \qquad r = (\zeta - 1)/(\zeta + 1), \tag{6a, b}$$

and $\zeta = Z/\rho_0 c_0$; note also that $p_1^-/p_1^+ = rT_r/T_i$. From equations (3) and (4) the transmission coefficients are related to the elements of the scattering matrix $[T]$ by

$$T_i = T_{11} + rT_{12}, \qquad T_r = (T_{21}/r) + T_{22}, \tag{7a, b}$$

showing that T_i and T_r are both functions of r, and hence of Z as well as of geometry, ρ_0, c_0, f and M.

Calculated, or measured, values, or if required the spectral estimates of $(T_i)_a$ and $(T_r)_a$ corresponding to Z_a and the reflection coefficient r_a, together with $(T_i)_b$ and $(T_r)_b$ with r_b, can then be used in equations (7a) and (7b) to give

$$T_{11} = [(T_i)_a r_b - (T_i)_b r_a]/[r_b - r_a], \qquad T_{12} = [(T_i)_b - (T_i)_a]/[r_b - r_a], \tag{8a, b}$$

$$T_{21} = r_s r_b[(T_r)_a - (T_r)_b]/[r_b - r_a], \qquad T_{22} = [(T_r)_b r_b - (T_r)_a r_a]/[r_b - r_a]. \tag{8c, d}$$

With an anechoic termination $Z_a = \rho_0 c_0$, so that p_2^- and hence r_a are both zero, one has

$$T_{11} = (T_i)_a, \qquad T_{12} = [(T_i)_b - (T_i)_a]/r_b, \qquad T_{21} = (p_1^-/p_2^+)_a, \qquad T_{22} = (T_r)_b - T_{21}/r_b. \tag{9a, b}$$

Equations (9) together show that measurements of $(T_i)_b$ and $(T_r)_b$ are always required for the evaluation of T_{12} and T_{22}, unless the transfer matrices possess the required reciprocity property. If such is the case, as is normal with electrical networks, then T_{12} is minus T_{21} and T_{22} is the complex conjugate of T_{11}.

Clearly, when a mean flow exists, the acoustic behaviour cannot be reciprocal, and similarly also when axial temperature gradients are present. Both conditions are relevant for IC engine exhaust systems, although it has been shown [3] that, with the axial gradients normally present, a close approximation to the acoustic behaviour is predicted if the average temperature over the system element is adopted to evaluate ρ_0 and c_0. It has also been shown [2] that, at the lower excitation frequencies of practical interest, neglect of wave attenuation and dispersion by viscothermal action may not provide valid predictions of acoustic performance. Finally, one must always ensure that all relevant boundary conditions are included during the analysis.

One can always check whether the transfer matrices for any particular system element possess the reciprocal property by seeing whether the value of the ratio $R_T = \mathrm{Re}\,(T_{22})/\mathrm{Re}\,(T_{11})$ is unity and whether $I_T = \mathrm{Im}\,(T_{22})/\mathrm{Im}\,(T_{11})$ is equal to -1. A similar check can be made with the values of the equivalent ratios R_c and I_c formed with T_{12} and T_{21} to see that both equal -1, although experience suggests that the first pair provide a sufficient and more sensitive test.

A simple cylindrical expansion chamber with hard walls and zero mean flow provides a first example, since it is widely accepted that its transfer matrix is reciprocal. However, it is only exactly so if the evanescent waves required to satisfy the boundary conditions over the end surfaces are neglected. Calculations performed with a four-fold area expansion ratio (chamber to pipe area equal to 4·17) and a chamber aspect ratio (length to diameter) of 6·25 showed that inclusion of the boundary conditions at the chamber end surfaces introduced systematic deviations from unity in the value of R_T, but insignificant changes in the corresponding values of the other three ratios.

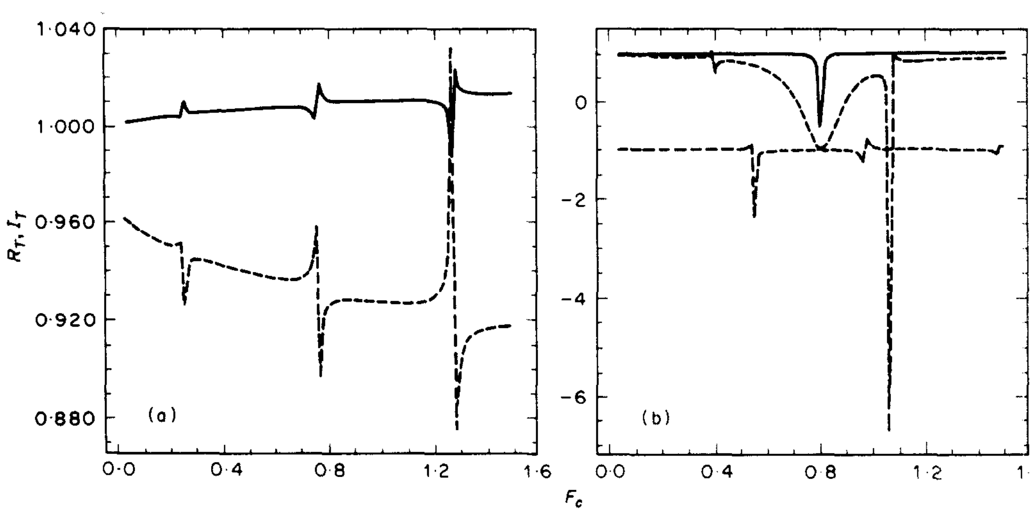

Figure 2. Acoustic transmission with different media: (a) for a simple expansion chamber; (b) including side branches of equal length. ——, R_T, with an ideal fluid; – – –, T_R; -- -- --, I_T, with air.

The upper curve in Figure 2(a) presents the results of the calculations of R_T covering a range of frequencies, f, corresponding to the first three acoustic wavelengths. They are plotted against the reduced frequency $F_c = L_c f/c_0$, with the chamber length $L_c = 0{\cdot}5$ m, a value adopted throughout the calculations reported here. The results indicate a systematic increase in the value of R_T with frequency, with distinct fluctuations at odd multiples of a half wavelength. The lower curve in Figure 2(a) demonstrates the result of including viscothermal effects in the calculation of R_T. In this case there is also a corresponding trend with frequency to higher negative values for the ratio I_T, the largest exceeding -2 at one wavelength. The corresponding values of the other two ratios R_c and I_c did not depart significantly from -1.

An expansion chamber which includes two side branches of equal length, so that the geometry remains symmetrical, provides a second example. It is well known that the acoustic transfer characteristics of such acoustic filters are dominated by the sequence of side branch resonances. The results of calculations with side branch lengths $L_s = 0{\cdot}15$ m, giving an effective first side branch resonance at $F_c = 0{\cdot}8$ when the appropriate boundary conditions are included [2], are plotted in Figure 2(b). The upper curve confirms that the value of R_T is strongly influenced by side branch resonance. The lower two curves give the corresponding results for R_T and I_T when viscothermal effects are included.

An expansion chamber which includes the practically more useful case of two unequal side branches with L_s equal to 0·2 and to 0·1, corresponding to first resonances at $F_c =$ 0·61 and 1·18 respectively, provides a third example. The total length of the side branches remains the same as in the previous example, while the calculations were made with viscothermal effects included to limit the amplitudes of the resulting peaks. In Figure 3(a) are shown the results for R_T and I_T, where the marked effect of first side branch resonance at $F_c = 0{\cdot}61$ interacting with the chamber resonance at $F_c = 0{\cdot}78$ can be seen. In this case the calculations of R_c and I_c are also included in Figure 3(b) where the influence of the first side branch resonances at $F_c = 0{\cdot}61$ and 1·18 are clearly shown.

The calculations for the first and third examples were repeated with a steady mean flow in the pipe corresponding to a Mach number of 0·14. The results for I_T and R_T for the expansion chamber, shown in Figure 4(a), were calculated both with an inviscid medium and with viscothermal effects included. Comparison with Figure 2(a) demonstrates the large changes in R_T and I_T results from the flow. There is now also a distinct influence on the values of I_T, as indicated in Figure 4(b). The influence of the same mean flow in the third example introduced distinct changes in the amplitudes and frequencies of the resonant peaks, as can be seen by comparing the results in Figures 5(a) and 5(b) with those in Figures 3(a) and 3(b). This apparently runs counter to some claims in the literature that the influence of mean flow on the side branch resonances may be neglected at low Mach numbers. In the present case the value of the Mach number within the chamber is less than 0·034, while the influence on the calculated ratios of the scattering matrix elements is obviously large. This arises mainly from the influence that the flow in the attached pipes has on the transfers at the junctions.

The results presented here show that the transfer characteristics of the representative acoustic filters considered here do not possess the reciprocity property unless the relevant boundary conditions are ignored, the fluid medium is inviscid, the geometry is symmetrical and there is no mean flow. All of these restrictions are hardly plausible in a practical

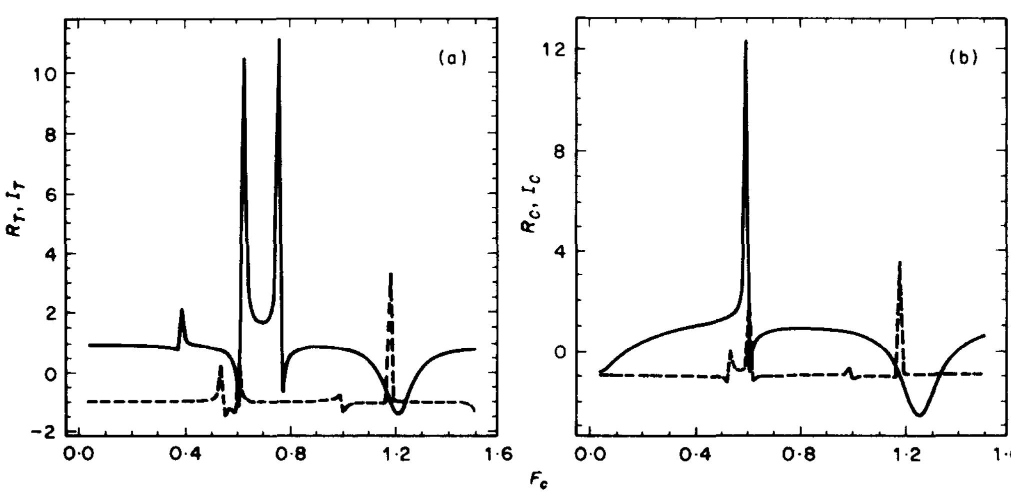

Figure 3. Acoustic transfer with dry air and side branches of differing length. (a) ——, R_T; - - -, I_T; (b) ——, R_c; -- -- --, I_c.

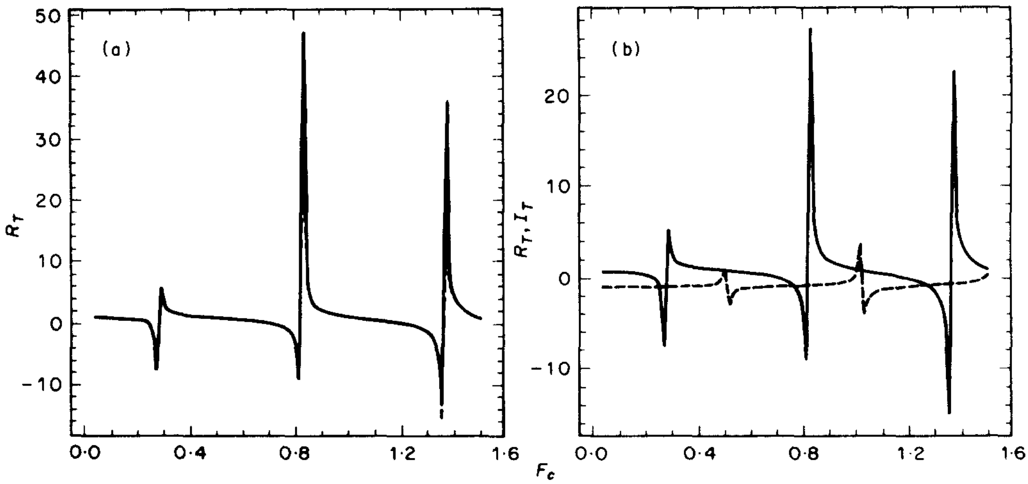

Figure 4. Acoustic transfer with a mean flow through a simple expansion chamber. (a) ——, R_T, ideal fluid; - - -, R_T, air; (b) ——, R_T, air; - - - -, I_T, air.

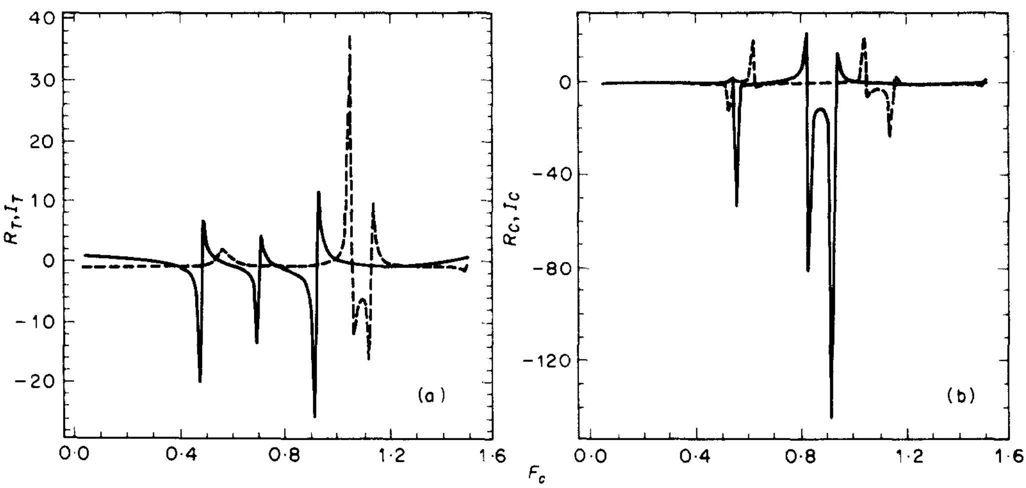

Figure 5. Acoustic transfer with a mean airflow and side branches of differing length. (a) ——, R_T; - - -, I_T; (b) ——, R_c; - - - -, I_c.

application. The rescaling of measured transmission matrices to account for changes in mean mass flow, in the temperature and in the other physical properties of the acoustic medium is also not directly obvious. Temperature changes can be approximately accommodated by rescaling the corresponding frequencies of excitation by the ratios of the corresponding sound speeds to maintain the factors fL_c/c_0 at a constant value, but this does not accommodate the accompanying changes in fluid density and viscosity.

It has been shown also that evaluation of the four components of the transmission matrices requires two measurements of the transmission coefficient T_i and T_r with distinctly different acoustic termination impedances Z. Thus, in all cases of parametric studies with a system, it is simpler to work directly with T_i and T_r, unless all the flow conditions along with the filter geometry remain invariant to ensure that the components of the transfer matrices retain constant values. The mathematical convenience arising from the fact that the components of the transfer matrix remain independent of r is of little practical value, due to this lack of flexibility, when the geometry of the system elements requires successive adjustments during system development towards a required acoustic performance. The results also demonstrate the care necessary to ensure that all acoustically relevant physical features of an acoustic filter are included in any evaluation of acoustic transfer characteristics. Finally, it is clear that this restriction will severely limit the validity of the electroacoustic analogy for this purpose, since electrical networks possess the reciprocal property while exhaust system acoustic elements generally do not.

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